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POLYNOMIAL STRUCTURES ON POLYCYCLIC GROUPS

KAREL DEKIMPE AND PAUL IGODT

Abstract. We know, by recent work of Benoist and of Burde & Grunewald, that there exist polycyclic–by–finite groups G, of rank h (the examples given were in fact nilpotent), admitting no properly discontinuous affine action on $\mathbb{R}^h$. On the other hand, for such G, it is always possible to construct a properly discontinuous smooth action of G on $\mathbb{R}^h$. Our main result is that any polycyclic–by–finite group G of rank h contains a subgroup of finite index acting properly discontinuously and by polynomial diffeomorphisms of bounded degree on $\mathbb{R}^h$. Moreover, these polynomial representations always appear to contain pure translations and are extendable to a smooth action of the whole group G.

1. Introduction.

In 1977 ([18]), John Milnor asked if it was true that any torsion-free polycyclic–by–finite group G occurs as the fundamental group of a compact, complete, affinely flat manifold. This is equivalent to saying that G acts properly discontinuously and via affine motions on $\mathbb{R}^{h(G)}$. (Throughout this paper, we will use $h(G)$ to denote the Hirsch length of a polycyclic–by–finite group G.) As there are several definitions of the notion "properly discontinuous action" we write down the definition we use:

Definition 1.1. Let Γ be a group acting (on the left) on a Hausdorff space X. We say that the action of Γ on X is properly discontinuous if and only if

1. $\forall x \in X$, there exists a neighbourhood U_x of x, such that ${}^{\gamma}U_x \cap U_x = \emptyset$ for all but finitely many $\gamma \in \Gamma$.
2. Let x and y be two elements in X, such that $x \notin \Gamma y$; then there exist neighbourhoods U_x and U_y of x and y resp. such that ${}^{\gamma}U_x \cap U_y = \emptyset$ for all $\gamma \in \Gamma$.

If the space X is locally compact (e.g. $X = \mathbb{R}^d$) this definition is equivalent to the requirement that for any compact subset $C \subseteq X$, ${}^{\gamma}C \cap C = \emptyset$ for all but a finite number of $\gamma \in \Gamma$.

It was shown by L. Auslander ([1]) in 1967, that any polycyclic–by–finite group G admits a faithful matrix representation (and so also an affine representation) into $\mathrm{SL}(n, \mathbb{Z})$ for some n. In his paper, J. Milnor showed that any virtually polycylic group G even acts properly discontinuously and effectively via affine transformations on some space $\mathbb{R}^n$. However, the n needed (both by Auslander and Milnor) was much bigger than the Hirsch length of G. Hence, Milnor's question stated above.

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A representation of G into $\mathrm{Aff}(\mathbb{R}^{h(G)})$ letting G act properly discontinuously on $\mathbb{R}^{h(G)}$ is called an **affine structure** on G. Milnor's question can now be formulated as follows: is it true that any polycyclic–by–finite group admits an affine structure? For a long time, only a few special cases, providing positive evidence, were known ([18], [20], [15], ...).

Recently however, Y. Benoist ([3], [4]) produced an example of a 10–step nilpotent (!) group of Hirsch length 11 which does not have an affine structure. This example was generalized to a family of examples by D. Burde and F. Grunewald ([6]). In ([5]) Burde even constructs counterexamples of nilpotency class 9 and Hirsch length 10.

Inspired by these counterexamples but also knowing that any torsion–free polycyclic–by–finite group acts smoothly and properly discontinuously on $\mathbb{R}^{h(G)}$ (see [7] and [14]), we started to study the existence of polynomial actions. Indeed, if we regard affine representations as being polynomial of degree ≤ 1 and smooth actions as being "polynomial of infinite degree" (think of a Taylor series expansion), then the investigation of polynomial actions is the next logical step. As a variation of Milnor's question, we then should ask:

> Does any (torsion–free) polycyclic–by–finite group G admit a properly discontinuous action on $\mathbb{R}^{h(G)}$, such that the action of each element $g \in G$ is expressed by a polynomial diffeomorphism of $\mathbb{R}^{h(G)}$ (and such that the degrees of these polynomials are bounded above).

We will refer to a properly discontinuous and polynomial action of a polycyclic–by–finite group G on $\mathbb{R}^{h(G)}$ as a **polynomial structure** on G. Remark that the quotient space $G\backslash\mathbb{R}^{h(G)}$ is automatically compact. If all polynomials involved are of degree $< m$, we say the polynomial structure is of degree $< m$. The results we have obtained so far ([11], [10] and [8]) lead to the following conjecture.

> **Conjecture[1]: Any polycyclic–by–finite group admits a polynomial structure.**

This paper is devoted to showing (in sections 2 and 3) that any polycyclic–by–finite group G has a subgroup of finite index (which one may assume to be characteristic in G) having a polynomial structure. Indeed, our main theorem can be stated as follows:

Theorem 1.2 (Main theorem)**.** *Let G be a polycyclic group such that $G/\operatorname{Fitt}(G)$ is free abelian. Then G acts properly discontinuously on $\mathbb{R}^{h(G)}$ via polynomials of degree $< \operatorname{Max}(h(G), 2)$.*

We also recall the notion of the affine defect number as introduced in [11]:

Definition 1.3. Let G be a polycyclic–by–finite group. The **affine defect of** G, denoted by $d(G)$, is defined as

$$\operatorname{Min}\left\{ s \in \mathbb{N} \,\Big\|\, \begin{array}{l} G \text{ acts properly discontinuously and via polynomial} \\ \text{diffeomorphisms of degree} \leq s+1 \text{ on } \mathbb{R}^{h(G)} \end{array} \right\},$$

[1] In the meantime, this conjecture was confirmed in ([9]), by means of a technique which is completely different from the one used in this paper. Although ([9]) provides the existence of polynomial structures in the general case, the present paper shows the existence of a much nicer (rather low degree) polynomial structure on a finite index subgroup. It is our conviction that a combination of this paper and ([9]) can lead to sharper results.

if there exists such an action for G; $d(G)=\infty$ if G does not allow any properly discontinuous action via polynomial diffeomorphisms of bounded degree on $\mathbb{R}^{h(G)}$.

As is easily seen, a torsion–free, polycyclic–by–finite group G has affine defect zero iff G occurs as the fundamental group of a compact, complete affinely flat manifold. Therefore, this affine defect number somehow measures the obstruction for G to be realized as the fundamental group of a compact, complete affinely flat manifold.

An alternative formulation for our main theorem is:

Let G be a polycyclic group such that $G/\operatorname{Fitt}(G)$ is free abelian. Then G has an affine defect $d(G)$ with $d(G) < \operatorname{Max}(h(G)-1, 1)$.

Remark 1.4. Let us remark here that a polynomial structure of degree 2 was obtained for the known counterexamples to Milnor's problem (i.e. the examples found in Burde & Grunewald), although the upper bound for this degree, according to the theorem, is 10. This indicates that the search for reducing this upper bound can be considered as a problem of major importance to think on.

In section 4, we pay attention to some properties of the polynomial structures obtained. It will be shown that these structures always contain pure translations (even central if possible). Furthermore, we describe how these polynomial structures can be considered as special cases of the Seifert fiber space construction (see [14]). This will allow us to make use of a uniqueness result of that theory, to prove that every polycyclic–by–finite group G allows a smooth action on $\mathbb{R}^{h(G)}$ which restricts to a polynomial structure on a finite index subgroup of G.

We frequently work with a finitely generated, torsion–free nilpotent group N. For short, we will refer to this situation by saying that $N \in \mathfrak{T}$, N is a $\mathfrak{T}$–group or N is $\mathfrak{T}$. We will also use $\mathcal{H}(X)$ to denote the group of self homeomorphisms of a topological space X.

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2. Mal′cev coordinates.

In this paper we will make use of two kinds of coordinates on a nilpotent Lie group L, which are essentially due to Mal′cev.

Let L be a simply connected, connected c–step nilpotent Lie group of dimension d. Following the original ideas of Mal′cev [17], we will see that there is a nice polynomial action of $L \rtimes \operatorname{Aut}(L)$ on $\mathbb{R}^d$. Let us denote the Lie algebra of L by $\mathfrak{g}$. As L is c–step nilpotent, all $(c+1)$–fold Lie brackets in $\mathfrak{g}$ are trivial, i.e. $[X_1, \dots, [X_c, X_{c+1}] \dots] = 0$, $\forall X_1, \dots, X_c, X_{c+1} \in \mathfrak{g}$.

We fix a basis $(A_1, A_2, \dots, A_d)$ in $\mathfrak{g}$, and so any element of $\mathfrak{g}$ has a unique coordinate $(x_1, x_2, \dots, x_d)$ with respect to this basis. The exponential map $\exp : \mathfrak{g} \to L$ is an analytical bijection between $\mathfrak{g}$ and L, and we denote the inverse of exp by log. The following definition is slightly more general than the original one of Mal′cev.

Definition 2.1. Let $g \in L$. The Mal′cev coordinate of the first kind, denoted by $m(g)$, of g (with respect to the basis $(A_1, A_2, \dots, A_d)$ of $\mathfrak{g}$) is the coordinate of $\log(g)$ with respect to the basis $(A_1, A_2, \dots, A_d)$.

These Mal′cev coordinates determine an analytical one to one correspondence

$$m : L \to \mathbb{R}^d$$

between L and $\mathbb{R}^d$. We say that L is equiped with a Mal′cev coordinate system of the first kind. Such coordinates have some nice properties which we state in the following lemma:

Lemma 2.2. *Let L be a simply connected, connected, c–step nilpotent Lie group, with Mal′cev coordinates of the first kind $m : L \to \mathbb{R}^d$. Then*

1. *There exists a polynomial map $p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, such that*
$$\forall g, h \in L : \; m(gh) = p(m(g), m(h)).$$
Moreover, p is of total degree $\leq c$, and of degree $\leq \mathrm{Max}(1, c-1)$ in the first (or last) d variables.
2. *If $\alpha : L \to L$ is a morphism of Lie groups, then there exists a linear map l, depending on α, such that*
$$\forall g \in L : \; m(\alpha(g)) = l(m(g)).$$

This lemma is proved in [17]; for more explanation we refer the reader to [8].

There is a natural action of $L \rtimes \mathrm{Aut}(L)$ on L which is given by

$$\forall g, h \in L, \forall \alpha \in \mathrm{Aut}(L) : \quad {}^{(g,\alpha)}h = g\alpha(h). \tag{1}$$

In Mal′cev coordinates of the first kind, this action is expressed by polynomials of limited degree.

Theorem 2.3. *Let L be a connected and simply connected nilpotent d–dimensional Lie group of nilpotency class c equiped with a Mal′cev coordinate system of the first kind. The coordinate expression for the action of $L \rtimes \mathrm{Aut}(L)$ is polynomial of degree $\leq \mathrm{Max}(1, c-1)$.*

Proof. Let (h, α) be any element of $L \rtimes \mathrm{Aut}(L)$. We have to show that $m({}^{(h,\alpha)}g) = q(m(g))$, where $q : \mathbb{R}^d \to \mathbb{R}^d$ is some polynomial map of degree $\leq \mathrm{Max}(1, c-1)$ (depending on h and α). By Lemma 2.2 we see that

$$m({}^{(h,\alpha)}g) = p(m(h), m(\alpha(g))) = p(m(h), l(m(g))),$$

which proves the theorem. □

From this theorem, one immediately deduces that $L \rtimes \mathrm{Aut}(L)$ acts on $\mathbb{R}^d$ via polynomials of degree $\leq \mathrm{Max}(1, c-1)$, in such a way that the action of L on $\mathbb{R}^d$ is simply transitive. This action, which is nothing else than the coordinate expression of the action of $L \rtimes \mathrm{Aut}(L)$ on L, is of course defined as follows (using the notations of Lemma 2.2):

$$\forall (g, \alpha) \in L \rtimes \mathrm{Aut}(L), \; \forall r \in \mathbb{R}^d : \; {}^{(g,\alpha)}r = m(g\alpha(m^{-1}(r))) = p(m(g), l(r)).$$

Remark 2.4. In [8], this theorem was used to show that any virtually c–step nilpotent group has an affine defect which is $\leq \mathrm{Max}(c-2, 0)$. This upper bound is not the best possible, for one knows that virtually 3–step nilpotent groups have affine defect $= 0$, while a close examination of the Campbell–Baker–Hausdorff formula (used in the proof of Lemma 2.2) shows that virtually 4–step nilpotent groups have affine defect ≤ 1.

We also need Mal′cev coordinates of the second kind. For this we assume that L has a lattice (i.e. uniform, discrete subgroup) N. The coordinates we use are again not exactly the same as the ones used originally by Mal′cev. Indeed, we work from an ascending central series while Mal′cev uses descending central series. Nevertheless, the proofs of all our statements can be found (subject to minor adjustments) in Mal′cev's paper ([17]).

Let N be any uniform and discrete subgoup of L. Consider an ascending central series

$$1 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_d = N \tag{2}$$

of N, with infinite cyclic factors N_i/N_{i-1} $(1 \leq i \leq d)$. It is well known that such a filtration always exists, as N is a $\mathfrak{T}$–group. Now, we choose d elements $n_1, n_2, \ldots, n_d \in N$ in such a way that N_i/N_{i-1} is generated by $n_i N_{i-1}$. We refer to the d–tuple $(n_1, n_2, \ldots, n_d)$ as a canonical basis associated to the central series (2). It follows that any element $n \in N$ can be written uniquely as a product

$$n = n_1^{z_1} n_2^{z_2} \ldots n_d^{z_d}, \;\; (z_1, z_2, \ldots, z_d) \in \mathbb{Z}^d.$$

Moreover, any element $l \in L$ is uniquely expressible as

$$l = n_1^{x_1} n_2^{x_2} \ldots n_m^{x_d}, \;\; (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d.$$

Here $n_i^{x_i}$, can be interpreted as being

$$n_i^{x_i} = \exp(x_i \log n_i).$$

We are now ready to define Mal′cev coordinates of the second kind:

Definition 2.5. Let $g \in L$. The Mal′cev coordinate of the second kind, denoted by $M(g)$, of g (with respect to the chosen central series (2) and a corresponding associated canonical basis) is determined by

$$M(g) = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \Leftrightarrow g = n_1^{x_1} n_2^{x_2} \ldots n_d^{x_d}.$$

Again we remark that such coordinates determine an analytical one–to–one correspondence between L and $\mathbb{R}^d$. Moreover, we can formulate a lemma analogous to Lemma 2.2, except for the fact that there is not such an interesting knowledge of the degrees of the polynomials involved; also an endomorphism of L is no longer expressed by a linear, but by a polynomial map.

3. The polynomial structure.

Let G be a polycyclic group such that $G/\operatorname{Fitt}(G)$ is free abelian. It is our intention to show that G admits a properly discontinuous action on $\mathbb{R}^{h(G)}$, which is expressed by means of polynomial functions of bounded degree in the coordinates of $\mathbb{R}^{h(G)}$.

In this section we will first make some small reductions, and then we will construct an action of G on $\mathbb{R}^{h(G)}$, using the semi–simple splitting of polycyclic groups as conceived by D. Segal ([13] and [22]). This action will turn out to be properly discontinuous and polynomial of bounded degree.

First reduction : Let $N = \operatorname{Fitt}(G)$. Suppose that $\tau(N)$ denotes the torsion subgroup of N. Then $\tau(N)$ is finite, and we consider the group $G' = G/\tau(N)$, which is $\mathfrak{T}$–by–(free abelian). Any properly discontinuous action $\rho' : G' \to \mathcal{H}(\mathbb{R}^d)$ gives rise to a properly discontinuous action ρ of G on $\mathbb{R}^d$ by defining $\rho(g) = \rho'(g\tau(N))$.

Therefore, it is enough to consider only polycyclic groups G which fit in a short exact sequence

$$1 \to N = \mathrm{Fitt}(G) \to G \to \mathbb{Z}^k \to 1, \tag{3}$$

where N is a $\mathfrak{T}$–group. (In fact, this is not really a reduction, because any properly discontinuous action of a polycyclic–by–finite group G on $\mathbb{R}^{h(G)}$ will factor through $G/\tau(\mathrm{Fitt}(G))$. See [11].)

Second reduction : Let G be a polycyclic group satisfying a short exact sequence (3), with $N \in \mathfrak{T}$. By Proposition 5.1 (part 1) of [13], we may assume that there exists a group $\tilde{G}$, containing G, such that

1. $[\tilde{G} : G] < \infty$;
2. $\tilde{N} = \mathrm{Fitt}(\tilde{G})$ is a $\mathfrak{T}$–group;
3. $\tilde{G}/\tilde{N} \cong G/N \cong \mathbb{Z}^k$;
4. $\tilde{G} = \tilde{N}\tilde{C}$, for some nilpotent subgroup $\tilde{C}$ of $\tilde{G}$.

Of course, any properly discontinuous action of $\tilde{G}$ on $\mathbb{R}^d$ induces one of G, simply by taking the restriction. Therefore, it will be enough to prove our main theorem in case G is a polycyclic group such that $N = \mathrm{Fitt}(G) \in \mathfrak{T}$, G/N is free abelian and G contains a nilpotent supplement for its Fitting subgroup N, i.e. G contains a subgroup C which is nilpotent and $G = \mathrm{Fitt}(G)C$.

Third reduction : We can even go a little bit further. Let G be a polycyclic group as above with a nilpotent supplement C for its Fitting subgroup. Conjugation by an element $c \in C$ in G induces an automorphism $c^* \in \mathrm{Aut}(N)$. We can lift this automorphism uniquely to the Mal′cev completion L of N. (Remark that we are always working with the Mal′cev completion, which is a real Lie group, while in [13] the rational Mal′cev completion is used. However, this is of no influence at all. Note also that in [13] and [22], the authors use only action on the right, while we prefer left actions. This is simply a matter of taste and does not affect the main ideas.) We denote the semi–simple part of c^* by c_s^*. It is known that $c_s^* \in \mathrm{Aut}(L)$, but c_s^* needn't be an automorphism of N. However, by Proposition 7.1 (part 2) of [13], there is a group $\tilde{N}$ of L, containing N as a subgroup of finite index, such that $c_s^* \in \mathrm{Aut}(\tilde{N})$, for all $c \in C$. Now we consider $\tilde{G} = \tilde{N}G$ (for the exact definition of $\tilde{G}$ see [13]). We have that:

1. $[\tilde{G} : G] < \infty$;
2. $\tilde{N} = \mathrm{Fitt}(\tilde{G})$ is a $\mathfrak{T}$–group;
3. $\tilde{G}/\tilde{N}$ $(\cong G/N)$ $\cong \mathbb{Z}^k$ for some k;
4. $\tilde{G} = \tilde{N}\tilde{C}$, for some nilpotent subgroup $\tilde{C}$ of $\tilde{G}$;
5. $c_s^* \in \mathrm{Aut}(\tilde{N})$ for all $c \in \tilde{C}$.

Therefore, we may restrict ourselves to polycyclic groups $\tilde{G}$, with the properties 2,3,4 and 5 above. Such a group $\tilde{G}$ is said to be a splittable group. From this last reduction our main theorem will follow once we proved:

Theorem 3.1. *A splittable polycyclic group G admits a polynomial structure, which is of degree $< h(G)$ if G is not the infinite cyclic group.*

The rest of this section is devoted to the proof of this theorem.

The name "splittable" is meaningful, since it follows from Theorem 1 of chapter 7 in [22] (or Proposition 7.1 of [13]) that G is splittable if and only if G admits a semi–simple splitting. We recall the definition and some of its immediate consequences.

Definition 3.2. Let G be a polycyclic group. A polycyclic group $\bar{G}$ containing G is a **semi–simple splitting** for G if the following hold:

1. $\bar{G} = M \rtimes T$ where $M = \mathrm{Fitt}(\bar{G}) \in \mathfrak{T}$ and T is free abelian;
2. T acts by semi–simple automorphisms on M;
3. $G \lhd \bar{G}$, $G \cap T = 1$ and $\bar{G} = MG = GT$;
4. $M = (M \cap G)C_M(T)$.

If G has a semi–simple splitting $\bar{G} = M \rtimes T$, then $N = \mathrm{Fitt}(G) = M \cap G$. It is also important to note that the Hirsch length of G and the Hirsch length of M coincide (this follows easily from the third requirement in the definition of a semi–simple splitting). Moreover, the group T acts faithfully on M. Therefore, we can regard G as being a subgroup of $M \rtimes \mathrm{Aut}(M)$. We list some useful properties concerning this semi–simple splitting in the following lemma:

Lemma 3.3. *Let G be a splittable group, with semi–simple splitting $\bar{G} = M \rtimes T$ and Fitting subgroup N. Using the notations of Definition 3.2, we have that*

1. *For any $m \in M$ there exist a unique $g_m \in G$ and a unique $t_m \in T$ such that $m = g_m t_m$. Moreover,*
   (a) *$\forall m_1, m_2 \in M :\ t_{m_1 m_2} = t_{m_1} t_{m_2}$;*
   (b) *$\forall m_1, m_2 \in M :\ g_{m_1} = g_{m_2} \Leftrightarrow m_1 = m_2$;*
   (c) *$G = \{g_m \parallel m \in M\}$;*
   It follows that $M/N \cong T$, and so $[M, M] \subseteq [\bar{G}, \bar{G}] \subseteq N$.
2. *For any $g \in G$ there exist a unique $m_g \in M$ and a unique $t_g \in T$ such that $g = m_g t_g$. Moreover,*
   (a) *$\forall g_1, g_2 \in G :\ t_{g_1 g_2} = t_{g_1} t_{g_2}$;*
   (b) *$\forall g_1, g_2 \in G :\ m_{g_1} = m_{g_2} \Leftrightarrow g_1 = g_2$;*
   (c) *$M = \{m_g \parallel g \in G\}$;*

Proof. We only prove the first claim of this lemma. The second one is obtained by reversing the role of M and G. As $M \subseteq \bar{G} = G \rtimes T$, it is obvious that any element $m \in M$ can be written as a unique product of the form $m = g_m t_m$, for some $g_m \in G$ and some $t_m \in T$. Moreover, t_m is the image of m under the homomorphism of groups $\varphi : M \to T$ which is the composition

$$M \xrightarrow{i_M} \bar{G} = G \rtimes T \xrightarrow{p} T,$$

where i_M denotes the inclusion of M in $\bar{G}$ and $p : G \rtimes T \to T$ is the projection onto T. So, $t_{m_1 m_2} = t_{m_1} t_{m_2}$ for all $m_1, m_2 \in M$. As the kernel of φ is $G \cap M = N$, there is an induced monomorphism $\bar{\varphi} : M/N \to T : m \mapsto t_m$, which is also an epimorphism, since $G \rtimes T = GM$. So, $M/N \cong T$. Suppose now that $g_{m_1} = g_{m_2}$ for some $m_1, m_2 \in M$; then

$$\begin{aligned} (g_{m_2} t_{m_2})^{-1} g_{m_1} t_{m_1} \in M \quad &\Rightarrow \quad t_{m_2}^{-1} g_{m_2}^{-1} g_{m_1} t_{m_1} \in M \\ &\Rightarrow \quad t_{m_2}^{-1} t_{m_1} \in M \cap T = 1 \\ &\Rightarrow \quad t_{m_1} = t_{m2} \\ &\Rightarrow \quad m_1 = g_{m_1} t_{m_1} = g_{m_2} t_{m_2} = m_2. \end{aligned}$$

Now, let $g \in G$. Then $g \in \bar{G} = M \rtimes T$ can be written as $g = mt$ for some $m \in M$ and $t \in T$ (in fact $m = m_g$ and $t = t_g$ of the second statement of the lemma). So $g = g_m t_m t$. As above, we can show that $t_m t \in G \cap T = 1$, or that $g = g_m$. We conclude that $G = \{g_m \parallel m \in M\}$.

Finally, we have to show that $[\bar{G},\bar{G}] \subseteq N$. Let $m_1, m_2 \in M$ and $t_1, t_2 \in T$. It suffices to show that $[m_1t_1, m_2t_2] \in N$. As M/N is abelian, $[m_1, m_2] \in N$. The proof now follows easily by using the fact that $m_i = n_ic_i$ $(i = 1, 2)$ for some $n_i \in N$ and $c_i \in C_M(T)$ and by remembering that T is abelian. □

We now have enough material to construct the polynomial structure of G on $\mathbb{R}^{h(G)}$. We will first construct the action (using Mal′cev coordinates of the first kind) and then prove that this action is properly discontinuous (using more or less Mal′cev coordinates of the second kind).

Denote by L the Mal′cev completion of M; then there is a canonical embedding of $M \rtimes \mathrm{Aut}(M)$ into $L \rtimes \mathrm{Aut}(L)$. Via this embedding we see that G $(\subseteq M \rtimes \mathrm{Aut}(M))$ acts on L (via the action (1)). Let us denote this action by $\rho_L : G \to \mathcal{H}(L)$, which is given by (for the notations see Lemma 3.3)

$$\forall g \in G, \forall l \in L : \rho_L(g)l = {}^{g}l = {}^{(m_g,t_g)}l = m_gt_g(l). \tag{4}$$

By Theorem 2.3, we can use a Mal′cev coordinate system of the first kind to interpret this action as being a polynomial action on $\mathbb{R}^{h(G)}$ (note that the dimension of L is equal to $h(M) = h(G)$). The polynomials used to express this action are of degree $\leq \mathrm{Max}(1, c-1)$, where c denotes the nilpotency class of M. This nilpotency class is of course $< h(M) = h(G)$, unless G is the infinite cyclic group. We conclude that a polynomial action $\rho_1 : G \to \mathcal{H}(\mathbb{R}^{h(G)})$ is obtained, such that $\forall g \in G$ the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{m} & \mathbb{R}^{h(G)} \\ {\scriptstyle \rho_L(g)} \downarrow & & \downarrow {\scriptstyle \rho_1(g)} \\ L & \xrightarrow{m} & \mathbb{R}^{h(G)} \end{array}$$

In other words, $m : L \to \mathbb{R}^{h(G)}$ is a G–equivariant diffeomorphism.

So far, we have constructed a polynomial action ρ_1 of G on $\mathbb{R}^{h(G)}$ (of the right degree). We still have to show that this action is properly discontinuous. Remark that ρ_1 determines a properly discontinuous action if and only if ρ_L does. To see that ρ_L is a properly discontinuous action, we will use Mal′cev coordinates of the second kind. Indeed, we will see that (with the right choice of Mal′cev coordinates of the second kind) there is a diffeomorphism $\delta : L \to \bar{L} \times \mathbb{R}^k$, where $\bar{L}$ is the Mal′cev completion of N. Moreover, the action ρ_2 of G on $\bar{L} \times \mathbb{R}^k$ for which δ is a G–equivariant map, i.e for which the diagram

$$\begin{array}{ccc} L & \xrightarrow{\delta} & \bar{L} \times \mathbb{R}^k \\ {\scriptstyle \rho_L(g)} \downarrow & & \downarrow {\scriptstyle \rho_2(g)} \\ L & \xrightarrow{\delta} & \bar{L} \times \mathbb{R}^k \end{array}$$

is commutative $\forall g \in G$, will turn out to be a properly discontinuous action. Note that it is possible to prove the proper discontinuity in a direct way, but the switch to another Mal′cev coordinate system will be used in the sequel of this paper too.

There exists an ascending central series for M, with infinite cyclic factors

$$1 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{h(G)} = M,$$

such that there is a number $i = h(N)$ with $M_i = N$. This can be seen as follows: If M is nilpotent of class c, then M has an ascending central series with torsion–free factor groups, which is given by the isolators of the commutator subgroups (see [19] or [22]):

$$1 = \sqrt{\gamma_{c+1}(M)} \subseteq \sqrt{\gamma_c(M)} \subseteq \cdots \subseteq \sqrt{\gamma_2(M)} \subseteq \sqrt{\gamma_1(M)} = M.$$

We know that $\sqrt{\gamma_2(M)} \subseteq N \subseteq M$ and that both $N/\sqrt{\gamma_2(M)}$ and M/N are torsion–free (see Lemma 3.3). This is enough to conclude that a suitable refinement of the above central series becomes a central series containing N as one of its terms and having infinite cyclic factor groups.

We fix such a central series and choose a canonical basis

$$(n_1, n_2, n_3, \dots, n_i, m_1, m_2, \dots, m_k)$$

($i + k = h(M) = h(G)$) associated to it, in such a way that the m_i's belong to $C_M(T)$. (In fact, this last requirement is not really necessary, but it simplifies life a bit).

So each element $l \in L$ can be written uniquely as a product

$$l = n_1^{x_1} \dots n_i^{x_i} m_1^{y_1} \dots m_k^{y_k},\ (x_1, \dots, x_i) \in \mathbb{R}^i,\ (y_1, \dots, y_k) \in \mathbb{R}^k.$$

The set of elements l for which the coordinates $y_1 = y_2 = \cdots = y_k = 0$ forms a Lie subgroup $\bar{L}$ of L, which can be seen as the Mal′cev completion of N. So, we can view L as the topological product $\bar{L} \times \mathbb{R}^k$, where an explicit diffeomorphism is given by

$$\delta : L \to \bar{L} \times \mathbb{R}^k : l = n_1^{x_1} \dots n_i^{x_i} m_1^{y_1} \dots m_k^{y_k} \mapsto (n_1^{x_1} \dots n_i^{x_i}, (y_1, \dots, y_k)).$$

This is useful to us, since we want to apply the following principle:

Lemma 3.4. *Let E be a group acting on a topological product space $X = X_1 \times X_2$. Suppose that the action of E on X is of the form*

$$\forall e \in E,\ \forall x_1 \in X_1,\ \forall x_2 \in X_2 :\ {}^e(x_1, x_2) = (f_e(x_1, x_2), h_e(x_2)),$$

where f_e is a map from $X_1 \times X_2$ to X_1 and h_e is a homeomorphism of X_2. Assume moreover that E has a normal subgroup $Q \lhd E$ acting only on the first component of X, i.e. $\forall q \in Q,\ \forall (x_1, x_2) \in X :\ {}^q(x_1, x_2) = (h'_q(x_1), x_2)$ for some homeomorphism h'_q of X_1. Then

1. *$h_e = h_{eq}$ for all $e \in E$ and $q \in Q$, so there is an induced homomorphism $\rho : E/Q \to \mathcal{H}(X_2) : eQ \mapsto \rho(eQ) = h_e$;*
2. *if Q acts properly discontinuously on X_1 and E/Q acts properly discontinuously on X_2 (via ρ), then E acts properly discontinuously on $X = X_1 \times X_2$.*

Of course, in our setting we take $X = L$, $X_1 = \bar{L}$ and $X_2 = \mathbb{R}^k$. The group G acts on L and has a normal subgroup N, acting only on the first component $X_1 = \bar{L}$ of X. This action of N on $\bar{L}$ is properly discontinuous, since N acts on $\bar{L}$ by left translations and N is discrete in $\bar{L}$.

The only thing left to show is that the induced action of G/N on $\mathbb{R}^k$ is properly discontinuous. Let g be an element of G; then g can be written uniquely as a product $m_g t_g$, where $m_g \in M$ and $t_g \in T$. We decompose m_g with respect to the canonical basis, say

$$m_g = n_1^{\alpha_1(g)} \dots n_i^{\alpha_i(g)} m_1^{\beta_1(g)} \dots m_k^{\beta_k(g)}, \quad \text{for some } (\alpha_1(g), \dots, \alpha_i(g)) \in \mathbb{Z}^i,\ (\beta_1(g), \dots, \beta_k(g)) \in \mathbb{Z}^k. \tag{5}$$

Remark that the map $\beta : G/N \to \mathbb{Z}^k : g \mapsto (\beta_1(g), \dots, \beta_k(g))$ is an isomorphism of groups.

We now compute that for $l = n_1^{x_1} \dots n_i^{x_i} m_1^{y_1} \dots m_k^{y_k}$, $(x_1, \dots, x_i) \in \mathbb{R}^i$, $(y_1, \dots, y_k) \in \mathbb{R}^k$),

$$\begin{aligned} {}^g l &= m_g \, t_g(l) \\ &= m_g \, t_g(n_1^{x_1} \dots n_i^{x_i}) \, m_1^{y_1} \dots m_k^{y_k} \\ &= \tilde{n} \, m_1^{\beta_1(g)+y_1} \dots m_k^{\beta_k(g)+y_k} \text{ for some } \tilde{n} \in N. \end{aligned}$$

This shows that G/N acts on $\mathbb{R}^k$, exactly as $\mathbb{Z}^k$ does via translations. This action is obviously properly discontinuous, and this concludes the proof of Theorem 3.1 and so also of our main theorem. □

Remark 3.5. Let G be a splittable polycyclic group with semi–simple splitting $\bar{G} = M \rtimes T$. The polynomial structure ρ_1 on G, as constructed above, is said to be the polynomial structure on G associated to the semi–simple splitting $\bar{G}$.

Note that another choice of Mal′cev coordinates of the first kind changes ρ_1, up to a conjugation with a linear diffeomorphism of $\mathbb{R}^{h(G)}$. This implies that the polynomial structure associated to a semi–simple splitting is also only determined up to a linear conjugation.

4. Some properties of the polynomial structure.

From now onwards, we restrict ourselves to splittable polycyclic groups. In view of the previous section, this is not really a restriction, since all our polynomial structures arise as polynomial structures on a splittable group G. We also write $Z(H)$ for the center of a group H.

It turns out that for the polynomial structures we construct there are always elements acting as pure translations. More precisely:

Theorem 4.1. *Let G be a splittable polycyclic group, with semi–simple splitting $\bar{G} = M \rtimes T$. Write $\rho_1 : G \to \mathcal{H}(\mathbb{R}^{h(G)})$ for the polynomial structure associated to $\bar{G}$. Then,*

1. *$Z(G)$ acts as pure translations on $\mathbb{R}^{h(G)}$;*
2. *there is always an element of G acting as a nontrivial pure translation on $\mathbb{R}^{h(G)}$.*

Proof. Let $z \in Z(G)$; then $z \in N = \text{Fitt}(G)$. We claim that $z \in Z(\bar{G})$. Indeed, consider any element $\bar{g}$ in $\bar{G}$. As $\bar{G} = G \rtimes T$, there is a unique decomposition $\bar{g} = gt$, for some $g \in G$ and $t \in T$. As z commutes with any element of G, we still have to show that z commutes with t. Let $C = C_G(T)$ denote the centralizer of T in G. If $\alpha \in \bar{G}$, we use α^* to refer to the automorphism of N induced by conjugation with α in $\bar{G}$, and α_s^* to indicate the semi–simple part of α^*. It is known (see [22, Lemma 10, page 141]) that $C_s^* = T^*$. Therefore, t acts on N in exactly the same way as c_s^* does, for some $c \in C$. But as $c^*(z) = z$, $\forall z \in Z(G)$, we can conclude that also $c_s^*(z) = z$, $\forall z \in Z(G)$ (e.g. use [22, Lemma 9, page 141]). This implies that

$$Z(G) \subseteq Z(\bar{G}) = Z(M \rtimes T) \subseteq Z(M).$$

However, it is obvious that the expression in Mal′cev coordinates of the first kind of the action of an element of $Z(M)$ on its Mal′cev completion L is given by a pure translation. This concludes the proof of our first statement.

Now, let c denote the nilpotency class of M. In the previous section we saw already that $\sqrt{\gamma_c(M)} \subseteq N \subseteq G$. But we also have that $\sqrt{\gamma_c(M)} \subseteq Z(M)$, so by the same argument as before we conclude that $\sqrt{\gamma_c(M)}$ is a nontrivial subgroup of G acting by pure translations on $\mathbb{R}^{h(G)}$. $\square$

Remark 4.2. This theorem is of interest, in regard to a conjecture of Auslander (see [2], [21]). The conjecture stated that any affine structure on a nilpotent group contained a nontrivial central translation. Although, in general, this turned out not to be the case (a counterexample is found in [12, section 3]), it is somehow remarkable that the (more general) polynomial structures obtained so far in our work always map the center of the group towards pure translations, and, in fact, always contain a nontrivial pure translation.

A logical step in an attempt to prove our conjecture might be to show that this polynomial structure on a splittable group G can be extended to any group containing G as a subgroup of finite index. In this perspective it is worthwhile to notice that the polynomial structure obtained so far can be viewed as a Seifert construction in the sense of [14].

We recall quickly the basic building blocks here, but refer to [14], for the details. Let L be any connected and simply connected nilpotent Lie group, and write W for any space. Then L acts on $L \times W$ as left translations on the first factor via: $l : L \to \mathcal{H}(L \times W)$, with $l(z)(x, w) = (zx, w)$ for all $z, x \in L$ and $w \in W$.

If W is a smooth manifold, $C(W, L^*)$ is used to denote the group of smooth mappings $\lambda : W \to L$, where the multiplication is (at first sight unnaturally) given by $\forall \lambda_1, \lambda_2 \in C(W, L^*)$, $\forall w \in W : (\lambda_1 * \lambda_2)(w) = \lambda_2(w)\lambda_1(w)$.

Still assuming that W is smooth, $\mathrm{Diff}(W)$ refers to the group of self diffeomorphisms of W. The group $\mathrm{Aut}(L) \times \mathrm{Diff}(W)$ acts on $C(W, L^*)$:

$$\forall(\alpha, \gamma) \in \mathrm{Aut}(L) \times \mathrm{Diff}(W),\ \forall \lambda \in C(W, L^*) :\ {}^{(\alpha,\gamma)}\lambda = \alpha \circ \lambda \circ \gamma^{-1}.$$

The resulting semi-direct product group $C(W, L^*) \rtimes (\mathrm{Aut}(L) \times \mathrm{Diff}(W))$, denoted by $\mathrm{Diff}^F(L \times W)$, embeds into $\mathrm{Diff}(L \times W)$ by defining

$$\forall \lambda \in C(W, L^*),\ \forall \alpha \in \mathrm{Aut}(L),\ \forall h \in \mathrm{Diff}(W),\ \forall x \in L,\ \forall w \in W :$$

$$(\lambda, \alpha, h)(x, w) = (\alpha(x)\lambda(h(w)), h(w)).$$

Remark that $\forall z \in L : l(z) = (z, \mu(z), 1) \in \mathrm{Diff}^F(L \times W)$.

A smooth set of data (N, E, ρ, W) for a Seifert construction consists of

1. a group E containg a normal subgroup N which is $\mathfrak{T}$, and
2. $\rho : E/N \to \mathrm{Diff}(W)$, a smooth action on a smooth manifold W.

Having a smooth set of data (N, E, ρ, W), the short exact sequence $1 \to N \to E \to Q \to 1$ induces a map $\psi : Q = E/N \to \mathrm{Out}(L)$, via conjugation in E. A smooth Seifert construction for this set of data is a homomorphism $\Psi : E \to \mathrm{Diff}^F(L \times W)$, making the following diagram commutative:

(6)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow l & & \downarrow \Psi & & \downarrow \psi \times \rho & & \\ 1 & \longrightarrow & C(W, L^*) \rtimes \mathrm{Inn}(L) & \longrightarrow & \mathrm{Diff}^F(L \times W) & \longrightarrow & \mathrm{Out}(L) \times \mathrm{Diff}(W) & \longrightarrow & 1 \end{array}$$

In [14], it was shown that, given a smooth set of data, their always exists a smooth Seifert construction, and, moreover, this Seifert construction is unique up to conjugation by an element of $C(W, L^*) \rtimes \mathrm{Aut}(L)$.

We are now ready to show the connection with the polynomial structures we constructed above.

Proposition 4.3. *Let G be a splittable polycyclic group, with semi–simple splitting $\bar{G} = M \rtimes T$. Use $\bar{L}$ to refer to the Mal′cev completion of* $\mathrm{Fitt}(G) = N$ *and let ρ_2 be the action (related to the polynomial structure associated to $\bar{G}$) of G on $\bar{L} \times \mathbb{R}^k$ as described in the previous section. If ρ denotes the induced action of G/N on $\mathbb{R}^k$ (by integral translations), then $(N, G, \rho, \mathbb{R}^k)$ is a smooth set of data for a Seifert construction for which ρ_2 is a Seifert construction.*

Proof. In the course of this proof, we will use $\mu(a)$ to denote conjugation with some element a in a group. Let $g \in G$; then g is uniquely expressible as a product $g = m_g t_g$, with $m_g \in M$ and $t_g \in T$. First we will show that $\rho_2(g) \in \mathrm{Diff}^F(\bar{L} \times \mathbb{R}^k)$. Consider any element $x \in \bar{L}$ and any element $y = (y_1, y_2, \dots , y_k) \in \mathbb{R}^k$. With the Mal′cev coordinate system of the second kind introduced in the previous section we have that $x = n_1^{x_1} n_2^{x_2} \dots n_i^{x_i}$. We compute

$$\begin{aligned}
\rho_2(g)(x, y) &= (\delta \circ \rho_L(g) \circ \delta^{-1})(x, y) \\
&= \delta({}^{(m_g,t_g)}(n_1^{x_1} \dots n_i^{x_i} m_1^{y_1} \dots m_k^{y_k})) \\
&= \delta(m_g t_g(n_1^{x_1} \dots n_i^{x_i}) m_g^{-1} m_g m_1^{y_1} \dots m_k^{y_k}) \\
&= \delta((\mu(m_g) \circ t_g)(x) m_g m_1^{y_1} \dots m_k^{y_k}) \\
&= (\alpha_g(x) \lambda_g(\gamma_g(y)), \gamma_g(y)),
\end{aligned}$$

where

1. $\alpha_g = \mu(m_g) \circ t_g \in \mathrm{Aut}(\bar{L})$;
2. $\gamma_g(y) = \rho(y) = (\beta_1(g) + y_1, \dots , \beta_k(g) + y_k)$ (see (5));
3. $\lambda_g \in C(\mathbb{R}^k, \bar{L}^*)$.

This shows that indeed $\rho_2(g) \in \mathrm{Diff}^F(\bar{L} \times \mathbb{R}^k)$. Moreover, if $g \in N$, then $\rho_2(g)$ acts on $\bar{L} \times \mathbb{R}^k$ via left translation on the first factor, from which we may conclude the commutativity of the left square in (6). To check the commutativity of the right square of (6) it is enough to realize that

$$\mu(g)_{|N} = (\mu(m_g) \circ t_g)_{|N},$$

which concludes the proof of our proposition. □

We can now use this result to prove that any polycyclic–by–finite group $\tilde{G}$ acts properly discontinuous on $\mathbb{R}^{h(\tilde{G})}$ in such a way that $\tilde{G}$ has a subgroup G of finite index acting via polynomial diffeomorphisms. We will have to use the fact that any $\tilde{G}$ has a characteristic and splittable subgroup G of finite index ([22, Theorem 2, page 145]).

Theorem 4.4. *Let $\tilde{G}$ be any polycyclic–by–finite group and denote by G any splittable characteristic subgroup of finite index, with semi–simple splitting $\bar{G} = G \rtimes T = M \rtimes T$. Then there exists a properly discontinuous and smooth action of $\tilde{G}$ on $\mathbb{R}^{h(\tilde{G})}$ such that the restriction of this action to G coincides with the polynomial structure of G associated to $\bar{G}$.*

Proof. Denote by N the Fitting subgroup of G. Then N is normal in $\tilde{G}$. It is possible to extend the action $\rho : G/N \to \mathcal{H}(\mathbb{R}^k)$ (by means of integral translations as discussed in the previous section) to an affine action $\tilde{\rho}$ of $\tilde{G}/N$ on $\mathbb{R}^k$ (e.g. see [16]). Then $(N, \tilde{G}, \tilde{\rho}, \mathbb{R}^k)$ is a smooth set of data for a Seifert construction. Let

$\tilde{\rho}_2 : \tilde{G} \to \mathrm{Diff}^F(\bar{L} \times \mathbb{R}^k)$, where $\bar{L}$ is the Mal′cev completion of N, denote a smooth Seifert construction for this set of data. Of course, the restriction $\tilde{\rho}_{2|G}$ of $\tilde{\rho}_2$ to G is a Seifert construction for the data $(N, G, \rho, \mathbb{R}^k)$. But by the previous theorem ρ_2 is also a Seifert construction for this set of data, so eventually, after conjugating $\tilde{\rho}_2$ with an element of $C(W, L^*) \rtimes \mathrm{Aut}(L)$, we may suppose that the restriction of $\tilde{\rho}_2$ to G is exactly ρ_2. Now, we define maps $\tilde{\rho}_L : G \to \mathcal{H}(L)$ and $\tilde{\rho}_1 : G \to \mathcal{H}(\mathbb{R}^{h(\tilde{G})})$ by requiring the commutativity of the diagram

$$\begin{array}{ccccc} \mathbb{R}^{h(\tilde{G})} & \stackrel{m}{\longleftarrow} & L & \stackrel{\delta}{\longrightarrow} & \bar{L} \times \mathbb{R}^k \\ {\scriptstyle \tilde{\rho}_1(\tilde{g})} \downarrow & & \downarrow {\scriptstyle \tilde{\rho}_L(\tilde{g})} & & \downarrow {\scriptstyle \tilde{\rho}_2(\tilde{g})} \\ \mathbb{R}^{h(\tilde{G})} & \stackrel{m}{\longleftarrow} & L & \stackrel{\delta}{\longrightarrow} & \bar{L} \times \mathbb{R}^k \end{array}$$

for all $\tilde{g} \in \tilde{G}$. It is now obvious that the restriction of $\tilde{\rho}_1$ to G equals ρ_1, the polynomial structure associated to $\tilde{G}$. This implies that $\tilde{\rho}_1$ is the desired action. □

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Katholieke Universiteit Leuven, Campus Kortrijk, B-8500 Kortrijk, Belgium
E-mail address: `Karel.Dekimpe@kulak.ac.be`

E-mail address: `Paul.Igodt@kulak.ac.be`